Announcements

1) HW #5 due Thursday

We were solving

$$y'' + (t-1)y' + y = 0.$$

We assumed $y(t) = \sum_{n=0}^{\infty} a_n t^n$

and got

$$(t-1)y' = -a_1 + \sum_{n=1}^{\infty} \left(na_n - (n+1)a_{n+1}\right) t^n$$

$$y'' = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n$$

$$0 = \overbrace{\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2}\, t^n}^{y''}$$

$$+ \overbrace{\sum_{n=0}^{\infty} a_n t^n}^{y} \overbrace{- a_1 + \sum_{n=1}^{\infty} \left(n a_n - (n+1)a_{n+1}\right) t^n}^{(t-1)y'}$$

Isolate t^0 term, combine all others:

$$0 = (2a_2 + a_0 - a_1) + \quad \text{bad!}$$

$$\sum_{n=1}^{\infty} (n+1)\left((n+2)a_{n+2} - a_{n+1} + a_n\right) t^n$$

Seems difficult to create a pattern from

$$2a_2 - a_1 + a_0 = 0$$

$$(n+1)\left((n+2)a_{n+2} - a_{n+1} + a_n\right) = 0$$

because there are three coefficients in each one.

Instead: assume

$$y(t) = \sum_{n=0}^{\infty} a_n (t-1)^n$$

on some nonzero radius of convergence about $c=1$.

Then $$y'(t) = \sum_{n=1}^{\infty} n a_n (t-1)^{n-1}$$

$$(t-1)y'(t) = \sum_{n=1}^{\infty} n a_n (t-1)^n$$

$$y''(t) = \sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (t-1)^n$$

So

$$0 = \overbrace{\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (t-1)^n}^{y''}$$

$$+ \underbrace{\sum_{n=1}^{\infty} n a_n (t-1)^n}_{(t-1)y'} + \underbrace{\sum_{n=0}^{\infty} a_n (t-1)^n}_{y}$$

Isolate $(t-1)^0$ coefficient, combine other terms:

$$0 = 2a_2 + a_0 + \sum_{n=1}^{\infty} (n+1)\left(a_{n+2}(n+2) + a_n\right)(t-1)^n$$

$$0 = 2a_2 + a_0 + \sum_{n=1}^{\infty} (n+1)\left(a_{n+2}(n+2) + a_n\right)(t-1)^n$$

If equal to zero, coefficients must equal zero, so

$$2a_2 + a_0 = 0, \quad a_2 = \frac{-a_0}{2}$$

$n=2$ $\quad \cancel{3}(4a_4 + a_2) = 0$

$$a_4 = \frac{-a_2}{4} = \frac{a_0}{2 \cdot 4}$$

$n=4$ $\quad \cancel{5}(6a_6 + a_4) = 0$

$$a_6 = \frac{-a_4}{6} = \frac{-a_0}{2 \cdot 4 \cdot 6}$$

Assuming a_0 is chosen,

$$\boxed{a_{2k} = \frac{(-1)^k a_0}{2^k k!}}$$

Assuming a_1 is chosen,

$$n=1 : \cancel{2}(3a_3 + a_1) = 0$$

$$a_3 = \frac{-a_1}{3}$$

$$n=3 \quad \cancel{4}(5a_5 + a_3) = 0$$

$$a_5 = -\frac{a_3}{5} = \frac{a_1}{3 \cdot 5}$$

$$n=5 : \quad 6(7a_7 + a_5) = 0$$

$$a_7 = \frac{-a_5}{7} = -\frac{a_1}{7 \cdot 5 \cdot 3}$$

$$n=7 \quad 8(9a_9 + a_7) = 0$$

$$a_9 = -\frac{a_7}{9} = \frac{a_1}{9 \cdot 7 \cdot 5 \cdot 3}$$

In general,

$$\boxed{a_{2k+1} = \frac{(-1)^k a_1 2^k k!}{(2k+1)!}}$$

$$a_7 \ (k=3) \quad \frac{\cancel{2} \cdot \cancel{2} \cdot \cancel{2} \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}{7 \cdot \cancel{6} \cdot 5 \cdot \cancel{4} \cdot 3 \cdot \cancel{2} \cdot \cancel{1}}$$

When do we know differential equations have power series solutions? (Section 8.3)

Analyticity: A function f is said to be analytic at $x=c$ if there are real numbers $a_0, a_1, a_2, \dots$ with

$$\boxed{f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n}$$

on some nonzero radius of convergence about $x=c$.

Example 2: (familiar functions)

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad (\text{radius} = \infty)$$

$$\sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \quad (\text{radius} = \infty)$$

$$\ln(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}(x-1)^n}{n} \quad (\text{radius} = 1)$$

This shows e^x, $\sin(x)$ are analytic at $c=0$ and $\ln(x)$ is analytic at $c=1$. In fact, e^x and $\sin(x)$ will be analytic for all values of c, but $\ln(x)$ will only be analytic for $x>0$.

Back to standard form

Given

$$a(t)y''(t)+b(t)y'(t)+c(t)y(t)=d(t),$$

divide by $a(t)$ and set

$$p(t)=\frac{b(t)}{a(t)},\ q(t)=\frac{c(t)}{a(t)}$$

$$r(t)=\frac{d(t)}{a(t)}.$$

$$\boxed{\text{Standard form: } y''+p(t)y'(t)+q(t)y(t)=r(t).}$$

Ordinary and Singular Points

We say $t=c$ is an ordinary point of $y'' + p(t)y' + q(t)$ if p, q are analytic at $t=c$ (except possibly at $t=c$).

We say $t=c$ is a singular point if one of p or q is not analytic at $t=c$.

Example 3: Find all singular points for

$$(x^2-4)y'' + (x-2)y' + (x^2-5x+6)y = 0$$

Divide by x^2-4:

$$y'' + \underbrace{\frac{x-2}{x^2-4}}_{p} y' + \underbrace{\frac{x^2-5x+6}{x^2-4}}_{q} y = 0$$

$$p(x) = \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2} \quad (x \neq 2)$$

$$p(x) = \frac{x-2}{(x-2)(x+2)} = \frac{1}{x+2} \quad (x \neq 2)$$

Only singular point is $x = -2$.

$$q(x) = \frac{x^2-5x+6}{x^2-4} = \frac{\cancel{(x-2)}(x-3)}{\cancel{(x-2)}(x+2)}$$

$$= \frac{x-3}{x+2}$$

except at $x = 2$.

Only singular point: $x = -2$

Regular and Irregular Singular Points

A singular point $x = c$ for $y'' + p(x)y' + q(x)$ is called regular if $(x-c)p(x)$ and $(x-c)^2 q(x)$ are analytic at $x = c$.

Otherwise, $x = c$ is called irregular

Example 4: For

$$y'' + \frac{x-2}{x^2-4}\,y' + \frac{x^2-5x+6}{x^2-4}\,y,$$

we found that $x = -2$

is a singular point.

Since $$\frac{x-2}{x^2-4} \cdot (x+2) = \frac{x-2}{x-2} = 1$$

and $$\frac{x^2-5x+6}{x^2-4} \cdot (x+2) = \frac{x^2-5x+6}{x-2}$$

$$= x-3$$

$(x \neq \pm 2)$

We get that $x = -2$ is a regular singular point since $f(x) = 1$ and $g(x) = x - 3$ are both analytic at $x = -2$.